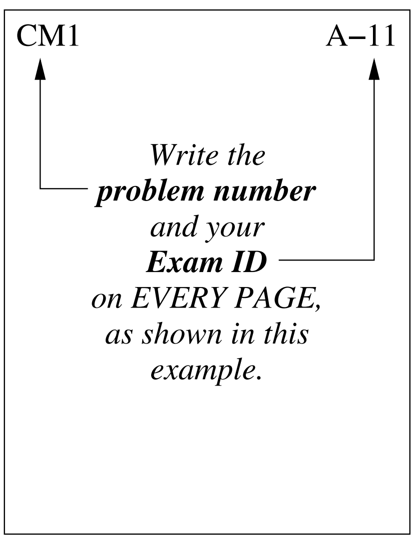
Department of Physics
Montana State University

Qualifying Exam
January, 2026

Day 1
Electricity and Magnetism



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(EM1) A point dipole with electric dipole moment $\mathbf{p} = p\,\hat{\mathbf{z}}$ is placed at the center of an **isolated, uncharged, conducting** spherical shell of inner radius a and thickness d. You are given several possibly non-zero terms of the potential inside the sphere (all higher order terms will be zero)

$$\varphi(r,\theta) = A + \frac{B}{r} + \left(Cr + \frac{D}{r^2}\right)\cos\theta$$

where (r,θ) are radius and polar angle in spherical coordinates. Use the boundary conditions, and $r \to 0$ asymptotic behavior to determine all coefficients in this expression. From it, find the distribution of induced charges on the sphere. Sketch the electric field inside and outside the shell.

Gradient in spherical coordinates is

$$\boldsymbol{\nabla} = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Solution:
Near the origin the dominant terms are the divergent terms

$$\varphi(r\to 0,\theta) = \frac{B}{r} + \frac{D}{r^2}\cos\theta$$

The first term is the Coulomb potential Q/r from charge Q at the origin and the second is the dipole term $\mathbf{p}\cdot\mathbf{r}/r^3$, and since there is no charge at the origin we set

$$B = Q = 0, \qquad D = p\,. \qquad \text{(Gaussian units)}$$

(One can also show that $B = 0$ by using Gauss law and integrating field flux through a spherical Gaussian surface centered at the origin. $0 = Q_{enclosed} = (1/4\pi)\iint E_r\, r^2 d\Omega = \iint (B/r^2 + (2D/r^3 - C)\cos\theta) r^2 d\Omega/4\pi = B$)

The metallic sphere has to be equipotential, so the tangential component of the field on the inner radius has to vanish:

$$E_\theta = -\frac{1}{r}\frac{\partial\varphi}{\partial\theta} = \left(C + \frac{D}{r^3}\right)\sin\theta\Big|_{r=a} = 0 \quad\Rightarrow\quad C = -\frac{D}{a^3} = -\frac{p}{a^3}\,.$$

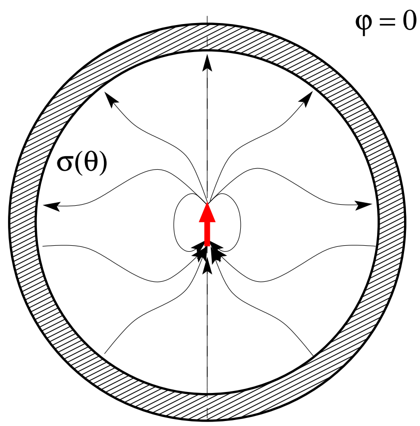
The field throughout the conducting material is zero; outside field is also zero, because the total charge enclosed is zero. The potential of the sphere is the same as that at infinity, which we take to be zero, and thus we set $A = 0$, giving us the answer for the potential

$$\varphi(r,\theta) = p\left(\frac{1}{r^2} - \frac{r}{a^3}\right)\cos\theta \qquad \text{for } r \leq a\text{; and zero everywhere else.}$$

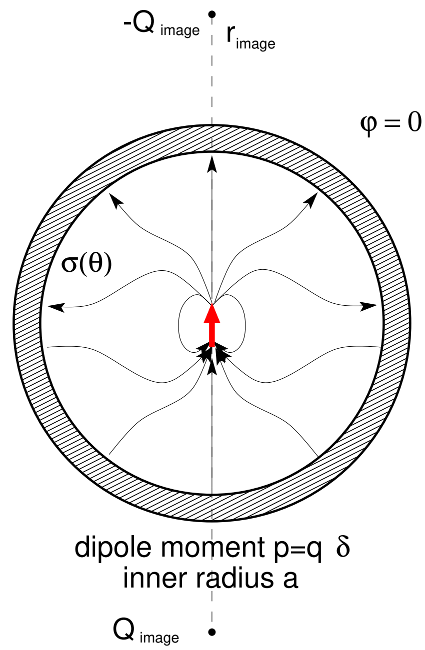
The dipole induces charges on the inner surface of the conducting shell, with the surface charge distribution given by (Gaussian units, the normal is in the negative radial direction)

$$\sigma(\theta) = \frac{E_n}{4\pi} = -\frac{E_r}{4\pi} = \frac{1}{4\pi}\frac{\partial\varphi}{\partial r}\Big|_{r=a} = -\frac{3p}{4\pi a^3}\cos\theta$$

The sketch of the field inside the sphere is shown in the figure. Field lines approach inner surface at the right angle. Total charge on the inner surface is zero, so there are no charges on the outside surface. Field outside is zero.



SOLUTION for potential $\varphi(r,\theta)$ using the method of images (as an example, not required for the exam, but closely related to solution of Laplace's equation).



We represent the dipole as two charges $(q/2, \delta)$ and $(-q/2, -\delta)$ with the property $\delta \to 0$ and $q \to \infty$, such that their product is finite:

$$\lim_{\delta \to 0} q\delta = p$$

To satisfy the boundary condition of constant potential on the sphere we introduce two image charges

$$Q_{image} = \frac{q}{2}\frac{a}{\delta}\,, \qquad r_{image} = \frac{a^2}{\delta}$$

and the total potential inside the sphere due to the four charges is

$$\varphi(r < a, \theta) = \frac{q/2}{\sqrt{\delta^2 + r^2 - 2r\delta\cos\theta}} - \frac{q/2}{\sqrt{\delta^2 + r^2 + 2r\delta\cos\theta}}$$
$$-\frac{Q_{image}}{\sqrt{r_{image}^2 + r^2 - 2rr_{image}\cos\theta}} + \frac{Q_{image}}{\sqrt{r_{image}^2 + r^2 + 2rr_{image}\cos\theta}}$$

Using expansion (generator of Legendre's polynomials)

$$\frac{1}{\sqrt{1 + x^2 - 2x\cos\theta}} = \sum_{\ell} x^{\ell} P_{\ell}(\cos\theta)$$

we write the sum as

$$\varphi(r<a,\theta)=\sum_{\ell}^{odd}\left(\frac{q\delta^{\ell}}{r^{\ell+1}}-\frac{2Q_{image}r^{\ell}}{r_{image}^{\ell+1}}\right)P_{\ell}(\cos\theta)=\sum_{\ell}^{odd}\left(\frac{q\delta^{\ell}}{r^{\ell+1}}-\frac{q\delta^{\ell}r^{\ell}}{a^{2\ell-1}}\right)P_{\ell}(\cos\theta)$$
$$=\left(\frac{p}{r^2}-\frac{pr}{a^3}\right)P_1(\cos\theta)$$

where in the last step we used that out of all the $q\delta^{\ell}$ terms, only $\ell=1$ term gives finite result $q\delta=p$, since all other terms vanish in $\delta\to 0$ limit.

(EM2) A charge-neutral, solid cylinder has radius R and height h. The material composing it obeys Ohm's Law and has uniform electrical conductivity σ. Within the cylinder is a uniform, time-dependent magnetic field $\boldsymbol{B}(t)$ directed along its axis.

(a) Calculate the *total rate* at which energy is dissipated in the cylinder due to its finite resistivity in the presence of a time-varying magnetic field.

(b) Describe a scenario whereby the experimentalist provides the energy that is dissipated in the material.

Solution:

(a) Ohm's Law is $\boldsymbol{J} = \sigma \boldsymbol{E}$, where $\boldsymbol{J}$ is the current density. The rate at which work is done per unit charge moving at velocity $\boldsymbol{v}$ is $\boldsymbol{v} \cdot \boldsymbol{E}$. For charge density ρ, the current density is $\boldsymbol{J} = \rho \boldsymbol{v}$. The work done per unit volume by the electric field is:

$$\frac{dW}{dV} = \boldsymbol{J} \cdot \boldsymbol{E} = \sigma E^2.$$

This is the dissipation rate (per volume) in the slab.

We need $\boldsymbol{E}$ induced by the changing magnetic field. From Faraday's Law:

$$\oint \boldsymbol{E} \cdot \boldsymbol{dl} = -\frac{d}{dt} \int \boldsymbol{B} \cdot \boldsymbol{dS}.$$

In order to choose a good Amperian loop, we need to know the direction of $\boldsymbol{E}$. Work in cylindrical coordinates (s, ϕ, z), where z is along the axis of the cylindrical slab. By Lenz' Law, an electric field will be induced in the slab that acts to oppose the change in flux through the slab; the induced electric field is axisymmetric, and in the $x - y$ plane. Since the system is charge neutral, $\nabla \cdot \boldsymbol{E} = 0$, and $\boldsymbol{E}$ has no radial component; hence, $\boldsymbol{E}$ is in the ϕ direction. A good Amperian loop is a circle of radius s.

From Faraday's Law:

$$E\, 2\pi s = -\dot{B} \pi s^2 \longrightarrow \boldsymbol{E} = -\frac{1}{2} \dot{B} s\, \hat{\phi}.$$

Lenz' Law and the right-hand rule confirm that the direction is correct. The total rate at which energy is being dissipated in the slab is:

$$W = \int_{\text{slab}} d^3r \, \frac{dW}{dV} = \int_{\text{slab}} d^3r \, \sigma E^2 = \sigma 2\pi h \int_0^R ds \, s E^2 = \frac{\pi}{8} \sigma \dot{B}^2 h R^4.$$

(b) To produce the external magnetic field, the experimentalist must drive currents with some sort of generator. To power the generator, the experimentalist must do work, say with a hand crank or a battery, at a rate W.

(EM3) A parallel plate capacitor is made of two conducting plates of area A, separated by $d \ll \sqrt{A}$. The capacitor is connected to a battery of voltage V_0, and allowed to fully charge. Then a dielectric slab of thickness $d/2$ and dielectric constant $\epsilon = 2\epsilon_0$ is inserted into the capacitor. This experiment is repeated in two different ways.

(a) In the first experiment, the battery is first **disconnected** and **then** the dielectric slab is inserted. Find the charge Q on the plates **and** the total electric field energy W_E, after the slab has been inserted.

(b) In the second experiment, the battery is left connected while the dielectric slab is inserted. Find Q and W_E for this case.

(c) In which configuration, (a) or (b), does the capacitor store more charge; in which configuration, does it store more electric field energy? Explain why.

Solution:
(a) Before the battery is removed, the electric field is $E = V_0/d$, and from Gauss law,

$$E = \frac{Q}{\epsilon_0 A},$$

so we find the total charge

$$Q = \epsilon_0 E A = \frac{\epsilon_0 A V_0}{d}.$$

After the battery is removed and slab placed in, Q remains constant. From Gauss' law, we find

$$D = \frac{Q}{A} = \frac{\epsilon_0 V_0}{d}.$$

The electric field in the vacuum and in the material of the capacitor is, therefore,

$$E_1 = \frac{D}{\epsilon_0} = \frac{V_0}{d}, \qquad E_2 = \frac{D}{2\epsilon_0} = \frac{V_0}{2d}.$$

So the potential difference between the two plates is

$$V = (E_1 + E_2)\frac{d}{2} = \frac{3}{4}V_0,$$

and the total electric field energy is

$$W_E = \frac{1}{2}QV = \frac{3\epsilon_0 A V_0^2}{8d}.$$

(b) Since the battery remains connected to the capacitor in this experiment, the potential difference between the two plates is fixed to V_0. If the total charge Q is transported to the capacitor, from Gauss law, we find $D = Q/A$. The electric field in the vacuum and material of the capacitor is

$$E_1 = \frac{D}{\epsilon_0} = \frac{Q}{\epsilon_0 A}, \quad E_2 = \frac{D}{2\epsilon_0} = \frac{Q}{2\epsilon_0 A}.$$

And the potential difference between the plates is therefore

$$V_0 = E_1\frac{d}{2} + E_2\frac{d}{2} = \frac{3}{4}Qd\epsilon_0 A,$$

leading to

$$Q = \frac{4\epsilon_0 A V_0}{3d}.$$

The total energy inside is therefore

$$W_E = \frac{1}{2}QV_0 = \frac{2\epsilon_0 A V_0^2}{3d}.$$

(c) Configuration in (b) has more charge Q and more energy W_E. In (a), when slab is placed in, electric field is doing work to polarize the material, so the total electric field energy is less. In (b), battery transports more charges to the capacitor to maintain V_0, and in this course, battery is doing more work and stores more electric field energy in the capacitor. Note that in both configurations, the capacitance of the capacitor is the **same** – this fact can also help you with solving part (b) quickly after solving (a). The capacitance only depends on the configuration and material of the capacitor, but does **not** depend on the voltage or charge applied on it. It is **not** correct to state that since (b) has more charge, the capacitance is larger.

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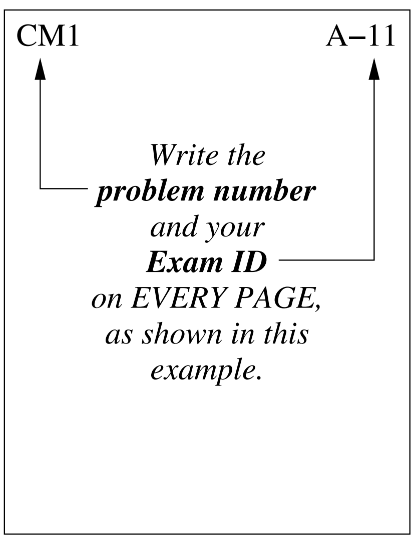
Day 2
Statistical and Thermal Physics



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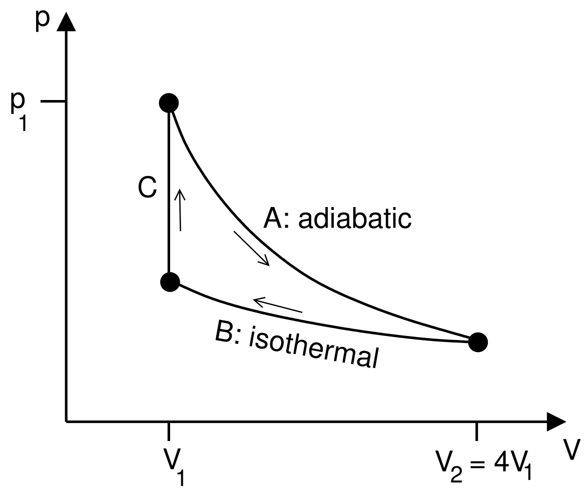
(ST1) A piston is filled with an ideal gas whose adiabatic index is $\gamma = 3/2$. The gas goes through a cycle composed of three stages, A, B, and C:

stage A: Beginning at pressure p_1 and volume V_1 the gas expands **adiabatically** to a volume $V_2 = 4V_1$.

stage B: The gas is **isothermally** compressed back to volume V_1.

stage C: The gas remains at volume V_1 while it returns to pressure p_1



Answer the following question about the cycle.

(a) Compute the total work **done by the gas** over the complete cycle. Express you result in terms of p_1 and V_1, being careful to assign the correct sign.

(b) Draw a graph of the cycle in S–T coordinates, with S as the horizontal axis. Label stages A, B, and C. Identify in which stage(s), if any, heat is **expelled from the gas**.

(c) Compute the heat **expelled** from the gas in stages identified in part (b). Express your result in terms of p_1 and V_1 **only**. You may wish to use the expression for the entropy of N particles of an ideal gas (up to an additive constant).

$$S = \frac{k_B N}{\gamma - 1} \ln\big(p V^{\gamma}\big) \quad . \tag{1}$$

Solution:

(a) Over stage A the pressure is given by $p \propto V^{-\gamma} = V^{-3/2}$. To account for the initial state, we may write it as

$$p_A(V) \;=\; p_1 \left(\frac{V}{V_1}\right)^{-3/2} \quad , \tag{2}$$

The stage ends at

$$p_2 = p_A(V_2) \;=\; p_1 \left(\frac{V_2}{V_1}\right)^{-3/2} \;=\; p_1\,(4)^{-3/2} \;=\; \frac{p_1}{8} \tag{3}$$

Stage B is isothermal, $p \propto V^{-1}$, so

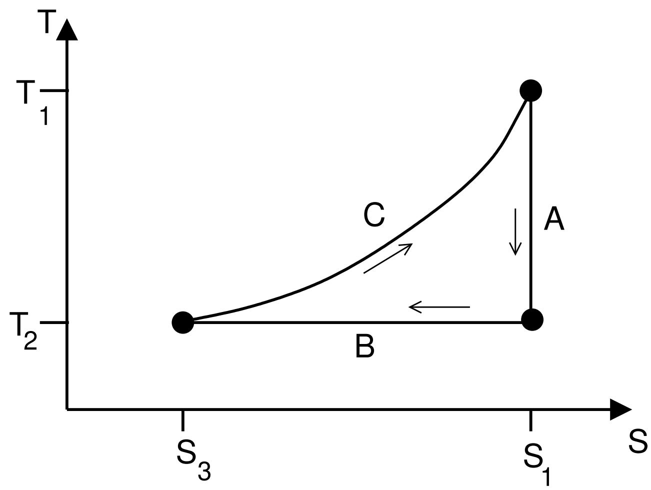
$$p_B(V) \;=\; p_2 \left(\frac{V}{V_2}\right)^{-1} \;=\; \frac{p_1}{8} \left(\frac{V}{4V_1}\right)^{-1} \;=\; \frac{p_1}{2} \left(\frac{V}{V_1}\right)^{-1} \tag{4}$$

The total work done **by the gas** is the negative of the work done **on the gas**

$$\begin{aligned}
W \;&=\; \oint p(V)\,dV \;=\; \int_{V_1}^{V_2} p_A(V)\,dV \;+\; \int_{V_2}^{V_1} p_B(V)\,dV \;+\; 0 \\
&=\; p_1\,V_1^{3/2} \int_{V_1}^{4V_1} V^{-3/2}\,dV + \tfrac{1}{2} p_1 V_1 \int_{4V_1}^{V_1} \frac{dV}{V} \;=\; -2\,p_1\,V_1^{3/2}\,V^{-1/2}\Big|_{V_1}^{4V_1} + \tfrac{1}{2} p_1 V_1 \ln(V)\Big|_{4V_1}^{V_1} \\
&=\; -2p_1\,V_1 \left(\tfrac{1}{2} - 1\right) \;-\; \tfrac{1}{2}\,p_1 V_1 \ln(4) \;=\; p_1\,V_1\big(\,1 - \ln 2\,\big)
\end{aligned} \tag{5}$$

The result is positive so the **gas does work**.

(b) The diagram is



where S_3 is the entropy at the end of stage B. The gas expels heat when S decreases. This occurs **only in stage B**.

(c) The heat lost by the gas in stage B is

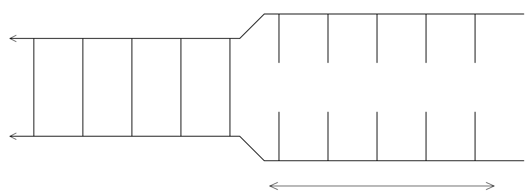
$$H \;=\; -\int_{S_1}^{S_3} T\,dS \;=\; -T_2\,(S_3 - S_1) \;=\; -2k_{\mathrm{B}}N\,T_2\,\ln\left(\frac{p_3\,V_1^{3/2}}{p_1\,V_1^{3/2}}\right)$$

$$=\; -2\,p_2\,V_2\,\ln\left(\frac{p_3}{p_1}\right) = -2\,\frac{p_1}{8}\,4V_1\,\ln\left(\frac{1}{2}\right) = p_1\,V_1\,\ln 2 \quad , \tag{6}$$

where p_3 is the pressure at the end of stage B, when $V = V_1$. Since stage B is isothermal, pV, is constant as $V_2 \to V_1$ and therefore $p_2V_2 = p_3V_1$, so

$$p_3 \;=\; p_2\,\frac{V_2}{V_1} \;=\; \frac{p_1}{8}\,\times\,4 \;=\; \frac{p_1}{2} \quad . \tag{7}$$

(ST2) An infinitely-long strand of DNA is in equilibrium with its environment at temperature T. The strand can unzip to some extent by first breaking a bond, of binding energy A, at one end. The strand can unzip further by breaking the next bond (also with binding energy A), and so on down the DNA strand. The strand will have, in general, N consecutive broken bonds counting from the right end of the strand, as shown:



(a) Determine the average number of bonds $\langle N \rangle$ that will be broken from the one end, and sketch this versus inverse temperature $\beta \equiv 1/k_B T$. Under what conditions does $\langle N \rangle$ become much larger than one?

(b) Determine the mean value of the size of the fluctuations in N about $\langle N \rangle$, in terms of βA alone. Describe how the size of the fluctuations, relative to $\langle N \rangle$, varies with temperature.

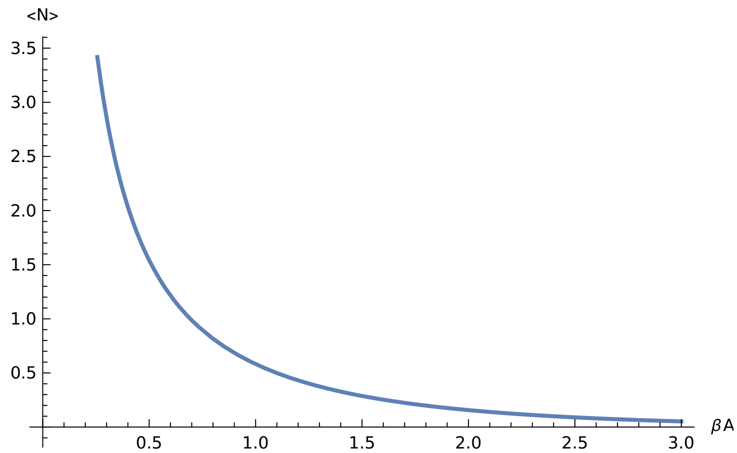
Solution:

(a) The partition function is:

$$Z = \sum_{N=0}^{\infty} e^{-N\beta A} = 1 + e^{-\beta A} + e^{-2\beta A} + ... = \frac{1}{1 - e^{-\beta A}}$$

The mean number of bonds that will break is:

$$\langle N \rangle = Z^{-1} \sum_{N=0}^{\infty} N e^{-\beta N A} = -\frac{1}{A}\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{1}{A}\frac{\partial \ln Z}{\partial \beta} = \frac{1}{e^{\beta A} - 1}$$



The number of broken bonds is large at high temperature such that $\beta A << 1 \to k_B T >> A$.

(b) The variance in N about the mean is:

$$
\begin{aligned}
(\Delta N)^2 &= \left\langle (N - \langle N \rangle)^2 \right\rangle = \left\langle N^2 \right\rangle - \langle N \rangle^2 \\
&= \frac{1}{A^2 Z} \frac{\partial^2 Z}{\partial \beta^2} - \left(\frac{1}{AZ} \frac{\partial Z}{\partial \beta} \right)^2 = \frac{1}{A^2} \frac{\partial}{\partial \beta} \left(\frac{1}{Z} \frac{\partial Z}{\partial \beta} \right) = -\frac{1}{A} \frac{\partial}{\partial \beta} \langle N \rangle \\
&= \frac{e^{\beta A}}{(e^{\beta A} - 1)^2} = \langle N \rangle^2 e^{\beta A}
\end{aligned}
$$

The size of fluctuations, relative to $\langle N \rangle$, is

$$\frac{\Delta N}{\langle N \rangle} = e^{\beta A/2},$$

which becomes exponentially large at low temperature (large βA), and small at high temperature.

(ST3) Consider a degenerate Fermi gas at zero temperature. The gas has N non-relativistic, non-interacting particles (each with mass m) and occupies a volume V. (The shape of this volume does not matter.) The degeneracy of each quantum state is g_s and is treated as a known constant.
(a) Find the Fermi energy ϵ_F of the gas in terms of its number density $n = N/V$. The Fermi energy is the energy of the highest occupied single-particle state when the gas is in its ground state.
(b) Find E_t, the total energy of the system in its ground state, and from it the average energy per particle E_t/N. How does E_t/N scale with the volume V?
(c) Find the pressure P of the ground state, in terms of n.
(d) Now consider a separate volume of a classical ideal gas with the same number density. What is the temperature required for the ideal gas to have the same pressure as the Fermi gas?

Solution:
(a). The number density of particles over phase space can be written as

$$n(p,x)d^3pd^3x = fg_s\frac{4\pi}{h^3}p^2dpdV, \tag{8}$$

where f is the occupation number per state and g_s is the degeneracy of each state ($g_s = 2$ for spin-1/2 electrons). For a zero-temperature Fermi gas, the occupation is simply given by

$$f = \begin{cases} 1, & \text{for } p \le p_F, \\ 0, & \text{for } p > p_F, \end{cases} \tag{9}$$

where p_F is the Fermi momentum and for non-relativistic particles, it is related to the Fermi energy as

$$\epsilon_F = \frac{p_F^2}{2m}. \tag{10}$$

Integrating over the phase space should give us the total number of electrons in the system, so we have the relation

$$\begin{aligned} N &= \int n(p,x)d^3pd^3x = 4\pi g_s\frac{V}{h^3}\int_0^{p_F} p^2dp, \\ &= \frac{4\pi}{3}g_s\frac{Vp_F^3}{h^3} = \frac{4\pi}{3}g_s\frac{V(2m_e\epsilon_F)^{3/2}}{h^3}. \end{aligned} \tag{11}$$

Thus,

$$\epsilon_F = \frac{h^2}{2m}\left(\frac{3}{4\pi g_s}n\right)^{2/3} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2}{g_s}n\right)^{2/3}. \tag{12}$$

Or in terms of p_F,

$$p_F = h\left(\frac{3}{4\pi g_s}n\right)^{1/3} = \hbar\left(\frac{6\pi^2}{g_s}n\right)^{1/3}. \tag{13}$$

(b). The total energy

$$\begin{aligned} E_t &= \int n(p,x)\frac{p^2}{2m}d^3p d^3x = \frac{4\pi g_s V}{2mh^3}\int_0^{p_F} p^4 dp, \\ &= \frac{4\pi g_s V}{5h^3}p_F^3\frac{p_F^2}{2m}, \\ &= \frac{3}{5}N\frac{p_F^2}{2m} = \frac{3}{5}N\epsilon_F. \end{aligned} \tag{14}$$

The averaged energy per particle is

$$\frac{E_t}{N} = \frac{3}{5}\epsilon_F \propto V^{-2/3} \tag{15}$$

(c). The most convenient way to get the pressure is to use the thermodynamic identity

$$P = -\left(\frac{\partial E_t}{\partial V}\right)_S = \frac{2}{3}\frac{E_t}{V} = \frac{2}{5}n\epsilon_F, \tag{16}$$

since S is conserved while the gas remains in its ground state.

(d). For an ideal gas, we have

$$P_{\text{ideal}} = \frac{N}{V}k_B T = nk_B T \tag{17}$$

For $P_{\text{ideal}} = P$, we thus need

$$k_B T = \frac{2}{5}\epsilon_F \propto n^{2/3}. \tag{18}$$