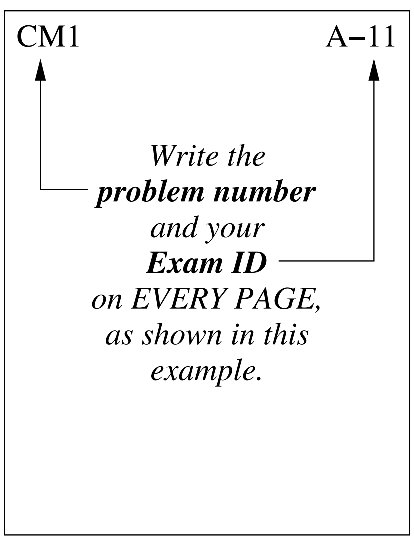
Department of Physics
Montana State University

Qualifying Exam
August, 2025

Day 1
Classical Mechanics



- Show your work.
- Write your solutions on the blank paper that is provided.
- Begin each problem on a new page. Write on only one side.
- If you do not attempt a problem, please turn in a blank sheet with your Exam ID and the problem number.
- Turn your work in to the proctor. There is a stack for each problem.
- Return all pages of this exam to the proctor, along with any writing that you do not wish to submit.

(CM1) A particle of mass m and energy E moves in the 1-d potential in the region $x > 0$:

$$V(x) = \frac{a}{x} + bx$$

where a and b are positive constants.

(a) Sketch the potential. Determine the range of motion in x and show this on your sketch. There is no friction.

(b) Determine the frequency of small oscillations about equilibrium. There is no friction.

(c) Now suppose the particle at velocity $\dot{x}$ experiences a frictional force $-\eta\dot{x}$. The particle is displaced a small distance from the above equilibrium and released. Determine the value of η for the subsequent motion to be *overdamped*, that is, with no oscillations.

(CM2) Consider a Lagrangian

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m\dot{q}^2 + \frac{GM\mu}{r} - \frac{1}{2}kq^2 + \Lambda\frac{q}{r^3}, \tag{1}$$

where (r, ϕ, q) are generalized coordinates and $(\dot{r}, \dot{\phi}, \dot{q})$ generalized velocities. Other quantities are constant parameters. Specifically, m, μ, M are some masses measured in [kg]. G is the gravitational constant.

(a) If r has the dimension of [length] and is measured in [m], what are the units of q, k, and Λ? Your result should involve only some combinations/powers of [kg], [m], and [s].

(b) Find the generalized momentum of each generalized coordinate. Are any of the momenta conserved?

(c) Find the total mechanical energy of the system up to an irrelevant constant. Is the total energy conserved?

(d) Show that the Lagrange's equation for r describes a particle moving in a 1-dimensional potential. Find the potential.

(CM3) A particle of mass m moves in two dimensions (x, y) in a trap with a potential:

$$V(x, y) = \alpha x y + \beta y^2$$

where α and β are positive constants.

(a) Obtain the Lagrangian and the equations of motion for the particle.

(b) Describe qualitatively the possible motion for $\alpha = 0$. No math should be necessary.

(c) Find the frequencies of the normal modes for oscillations about the origin, and the associated motion. Can the particle escape the trap, that is, move away from the origin without returning?

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Day 2
Quantum Mechanics

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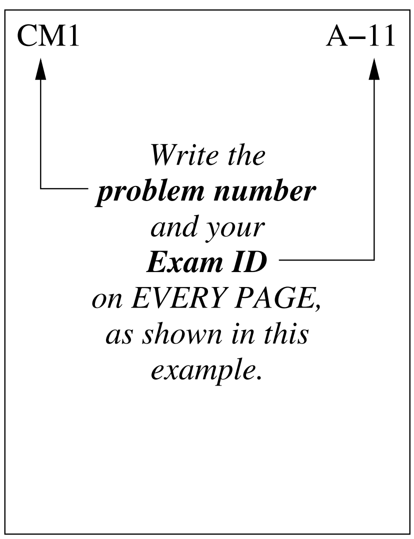


(QM1) A particle in some system can be in two eigenstates, with energies $E_1,\ E_2$ and the corresponding wavefunctions

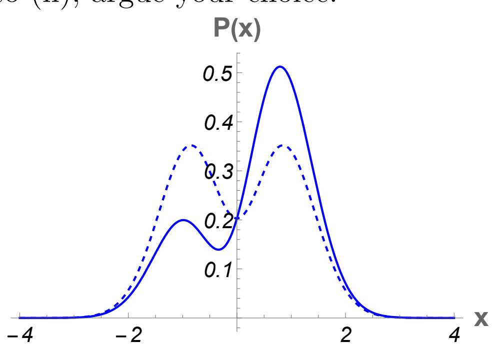
$$\psi_1(x) = \frac{1}{\pi^{1/4}}\, e^{-x^2/2} \qquad \text{and} \qquad \psi_2(x) = \frac{1}{(\pi/4)^{1/4}}\, x\, e^{-x^2/2}$$

where $x \in (-\infty, +\infty)$ is the dimensionless position.

One prepares 10 000 particles, each in an identical state $\psi(x)$, measures their energies and finds that about 3 600 of these particles have energy E_1 and the rest have energy E_2. From another experiment, it is determined that the relative phase between these energy states is $\theta = \pi/3$. We will neglect time evolution of the state, i.e. we'll keep θ fixed to this value.

We are interested in measurement of a particle's position in this system. Consider two cases: (i) the position is measured *before* the energy is measured; and (ii) the position is measured *after* the energy is measured.

(a) Below is the plot of probability density $P(x)$ of finding a particle in position x. Identify which graph (solid or dashed) corresponds to case (i) and which to (ii), argue your choice.



(b) Find the expectation value of the position in each of the two cases.

Useful integrals (integer $n = 0, 1, 2\dots$):

$$\int_{-\infty}^{+\infty} dx\, x^{2n} e^{-x^2} = \sqrt{\pi}\, \frac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2^n} \qquad (= \sqrt{\pi}\ \text{ if } n=0)$$

(QM2) Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be eigenstates of an operator $\hat{S}_z$ with eigenvalues $\pm\frac{1}{2}\hbar$. The spin operators satisfy the commutation relations

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y. \tag{2}$$

(a) Define $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$. Find the commutation relations $[\hat{S}_z, \hat{S}_\pm]$.

(b) Show that $\hat{S}_+|\downarrow\rangle$ is an eigenstate of $\hat{S}_z$. What is the eigenvalue?

(c) Starting with $\hat{S}_+|\uparrow\rangle = 0$, find $\hat{\mathbf{S}}^2|\uparrow\rangle$. (Hint: relate $\hat{\mathbf{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ to $\hat{S}_-\hat{S}_+$)

(QM3) A particle of mass m is in a square well potential with infinite walls at $x = 0$, $x = a$

$$V(x) \;=\; \begin{cases} 0 & , \quad 0 \;<\; x \;<\; a \\ \infty & , \quad \text{otherwise} \end{cases} \tag{3}$$

The energies and eigenfunctions are

$$E_n \;=\; \frac{\hbar^2\pi^2}{2ma^2}\,n^2 \quad , \quad \phi_n(x) \;=\; \sqrt{\frac{2}{a}}\,\sin\left(\frac{\pi x}{a}\,n\right) \quad , \quad n \;=\; 1,\,2,\,\ldots \tag{4}$$

The particle is subject to a perturbation

$$V'(x) \;=\; -\alpha\,\delta(x - a/3) \;\;, \tag{5}$$

where $\delta(z)$ is the Dirac delta function and $\alpha > 0$ is real. The perturbation is small in the sense that $\alpha \ll aE_1$.

(a) Compute the **first order** energy correction to the lowest two eigenstates

(b) Using **only the lowest two eigenstates**, write down the **first order** correction to the **ground state** wave function. Use this to write the ground state wave function as an **explicit function** of x. Ensure that your wave function is normalized to first order in the perturbation.

(c) Use the approximate wave function from (b) to compute the probability of finding the ground state particle within the left half of the well $0 < x < a/2$. You may express the result to **first order in the perturbation**.

The following anti-derivative may prove useful

$$\int \sin(bx)\,\sin(cx)\,dx \;=\; \frac{\sin[\,(b-c)x\,]}{2(b-c)} \;-\; \frac{\sin[\,(b+c)x\,]}{2(b+c)}$$