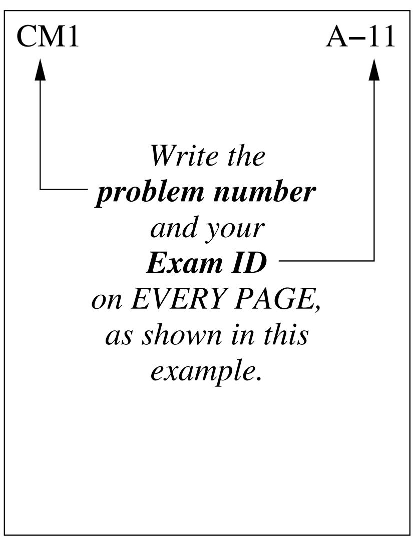
Department of Physics
Montana State University

Qualifying Exam
August, 2025

Day 1
Classical Mechanics



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(CM1) A particle of mass m and energy E moves in the 1-d potential in the region $x > 0$:

$$V(x) = \frac{a}{x} + bx$$

where a and b are positive constants.

(a) Sketch the potential. Determine the range of motion in x and show this on your sketch. There is no friction.

(b) Determine the frequency of small oscillations about equilibrium. There is no friction.

(c) Now suppose the particle at velocity $\dot{x}$ experiences a frictional force $-\eta\dot{x}$. The particle is displaced a small distance from the above equilibrium and released. Determine the value of η for the subsequent motion to be *overdamped*, that is, with no oscillations.

Solution:

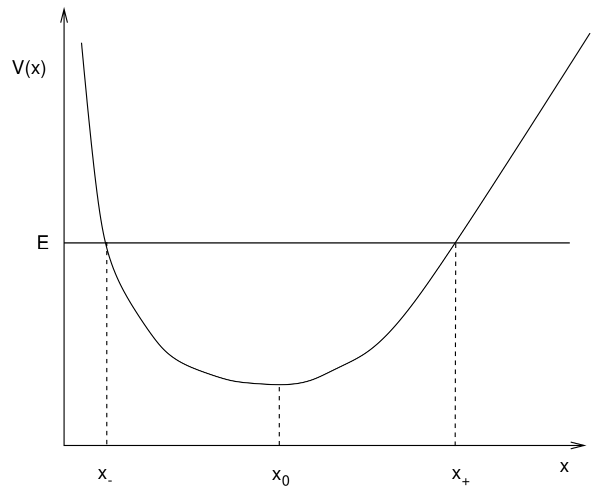
a) The single minimum of the potential is at $x_0 = \sqrt{a/b}$. The turning points are given by $V(x) = E$:

$$x_\pm = \frac{E}{2b} \pm \sqrt{\left(\frac{E}{2b}\right)^2 - \frac{a}{b}}$$

For there to be real solutions, E must satisfy:

$$E \geq E_{min} = 2\sqrt{ba}$$

If $E = E_{min}$, both turning points are at x_0; the particle stays at rest x_0. For $E > E_{min}$, the particle oscillates (though anharmonically) between the turning points. The greater the energy, the greater the distance between the two turning points.



b) The only equilibrium is $x_0 = \sqrt{a/b}$ as found in part (a). Expanding $V(x)$ about x_0 gives

$$V(x) = V(x_0) + \cancelto{0}{V'}(x_0)\,(x - x_0) + \frac{1}{2}V''(x - x_0)^2 + ...$$

The equation of motion, to leading order in $(x - x_0)$, is

$$m\ddot{x} = -V'(x) = -V''(x_0)(x - x_0)$$

or, in terms of $\delta x \equiv x - x_0$,

$$m\,\delta\ddot{x} + V''(x_0)\,\delta x = 0$$

which is the equation of an oscillator of frequency ω_0, given by

$$\omega_0^2 = \frac{V''(x_0)}{m} = \frac{2a}{mx_0^3} = \frac{2}{m}\left(\frac{b^3}{a}\right)^{1/2}$$

c) The equation of motion becomes:

$$\delta\ddot{x} + \frac{\eta}{m}\delta\dot{x} + \omega_0^2\delta x = 0$$

Seek a solution $\delta x \propto e^{i\omega t}$, giving,

$$-\omega^2 + \frac{i\eta}{m}\,\omega + \omega_0^2 = 0$$

There will be no oscillations if ω is imaginary. This happens for:

$$\eta \geq 2m\omega_0$$

Equality is the case of critical damping.

(CM2) Consider a Lagrangian

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m\dot{q}^2 + \frac{GM\mu}{r} - \frac{1}{2}kq^2 + \Lambda\frac{q}{r^3}, \tag{1}$$

where (r, ϕ, q) are generalized coordinates and $(\dot{r}, \dot{\phi}, \dot{q})$ generalized velocities. Other quantities are constant parameters. Specifically, m, μ, M are some masses measured in [kg]. G is the gravitational constant.

(a) If r has the dimension of [length] and is measured in [m], what are the units of q, k, and Λ? Your result should involve only some combinations/powers of [kg], [m], and [s].

(b) Find the generalized momentum of each generalized coordinate. Are any of the momenta conserved?

(c) Find the total mechanical energy of the system up to an irrelevant constant. Is the total energy conserved?

(d) Show that the Lagrange's equation for r describes a particle moving in a 1-dimensional potential. Find the potential.

Solution:

(a). In physics, only quantities with the same dimension/units can be summed. Since μ is in [kg] and r in [m], the $\mu\dot{r}^2$ term in $\mathcal{L}$ has units of $[\mathrm{kg\,m^2\,s^{-2}}]$. For $m\dot{q}^2$ to have the same units with m in [kg], q must be in [m]. Then k must have $[\mathrm{kg\,s^{-2}}]$ and Λ $[\mathrm{kg\,m^4\,s^{-2}}]$. In summary

$$q \sim [\mathrm{m}], \quad k \sim [\mathrm{kg\,s^{-2}}], \quad \Lambda \sim [\mathrm{kg\,m^4\,s^{-2}}]. \tag{2}$$

(b) We have $p_i = \partial\mathcal{L}/\partial\dot{q}_i$, so

$$p_r = \frac{\partial\mathcal{L}}{\partial\dot{r}} = \mu\dot{r}, \tag{3}$$

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \mu r^2\dot{\phi}, \tag{4}$$

$$p_q = \frac{\partial\mathcal{L}}{\partial\dot{q}} = m\dot{q}. \tag{5}$$

Since ϕ is a cyclic coordinate, p_ϕ is conserved.

(c) The total mechanical energy can be found by

$$
\begin{aligned}
E &= \sum_i p_i \dot{q}_i - \mathcal{L} \\
&= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{1}{2}m\dot{q}^2 - \frac{GM\mu}{r} + \frac{1}{2}kq^2 - \Lambda\frac{q}{r^3}.
\end{aligned} \tag{6}
$$

Alternatively, one may recall that $\mathcal{L} = T - U$. Simply flipping the signs of the potential terms (those independent of $\dot{q}_i$) in $\mathcal{L}$ would be sufficient.

The total energy is conserved because $\mathcal{L}$ does not explicitly depend on t. More rigorously, as most generically, $\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t)$, we have

$$
\begin{aligned}
\dot{E} &= \sum_i \left(\dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i - \frac{\partial \mathcal{L}}{\partial t} \right) \\
&= \sum_i \left\{ \left[\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}) - \frac{\partial \mathcal{L}}{\partial q_i} \right] \dot{q}_i + \left[p_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right] \ddot{q}_i - \frac{\partial \mathcal{L}}{\partial t} \right\}.
\end{aligned} \tag{7}
$$

In the second line, terms inside the bracket vanish from the definition of generalized momenta and the Lagrange's equation. Therefore,

$$
\dot{E} = -\frac{\partial \mathcal{L}}{\partial t}. \tag{8}
$$

(d). From the Lagrange's equation for r, we have

$$
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = \mu\ddot{r} = \frac{\partial \mathcal{L}}{\partial r} = \mu r\dot{\phi}^2 - \frac{GM\mu}{r^2} - \frac{3\Lambda q}{r^4} \tag{9}
$$

$$
= \frac{p_\phi^2}{\mu r^3} - \frac{GM\mu}{r^2} - \frac{3\Lambda q}{r^4} = -\frac{dU_{\text{eff}}}{dr}, \tag{10}
$$

where

$$
U_{\text{eff}}(r) = -\frac{G\mu M}{r} + \frac{p_\phi^2}{2\mu r^2} + \frac{\Lambda q}{r^3}. \tag{11}
$$

Note $\dot{\phi}$ is not a constant of motion in general, and it needs to be replaced by $\dot{\phi} = p_\phi/(\mu r^2)$.

(CM3) A particle of mass m moves in two dimensions (x, y) in a trap with a potential:

$$V(x, y) = \alpha xy + \beta y^2$$

where α and β are positive constants.

(a) Obtain the Lagrangian and the equations of motion for the particle.

(b) Describe qualitatively the possible motion for $\alpha = 0$. No math should be necessary.

(c) Find the frequencies of the normal modes for oscillations about the origin, and the associated motion. Can the particle escape the trap, that is, move away from the origin without returning?

Solution:

(a) The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2}m\omega_0^2 y^2 - \alpha xy$$

where $\omega_0^2 = 2\beta/m$. The Euler-Lagrange equations are

$$\ddot{x} + \frac{\alpha}{m} y = 0$$
$$\ddot{y} + \omega_0^2\, y + \frac{\alpha}{m} x = 0$$

(These equations also follow directly from Newton's Second Law.)

(b) For $\alpha = 0$, the particle moves at constant speed in the x direction while executing oscillations in the y direction of frequency ω_0.

(c) The equations of motion are linear, so we can seek a solution $\propto e^{i\omega t}$:

$$\begin{pmatrix} -\omega^2 & \alpha/m \\ \alpha/m & \omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0 \qquad (1)$$

The determinant of the matrix must vanish, giving,

$$\omega_\pm^2 = \frac{\omega_0^2}{2} \pm \left[\left(\frac{\omega_0^2}{2} \right)^2 + \frac{\alpha^2}{m^2} \right]^{1/2}$$

There are two kinds of solutions:

1) ω_+^2 is positive, so ω_+ is real with values $\omega_+ = \pm|\omega_+|$. These solutions are oscillatory with $|\omega_+| > \omega_0$.

2) ω_-^2 is negative, so ω_- is imaginary with values $\omega_- = \pm i|\omega_-|$. The root $-i|\omega_-|$ is an unstable mode, growing exponentially in time, while the root $-i|\omega_-|$ is a decaying mode.

Substituting the roots in eq. 1 gives

$$x(t) = \frac{\alpha}{m\omega_\pm^2}\, y(t)$$

For the ω_+ (oscillatory) modes, the particle moves at difference speeds in the x and y directions (recall that α is positive), with both coordinates changing in phase. For the ω_- modes, the coordinates are 180° out of phase. The motion in either case is along a line with slope

$$m_\pm = \frac{m\omega_\pm^2}{\alpha}.$$

For a small peturbation about the origin, the unstable mode will drive the particle away from the origin along the line of slope m_- (< 0), and the particle will escape the trap.

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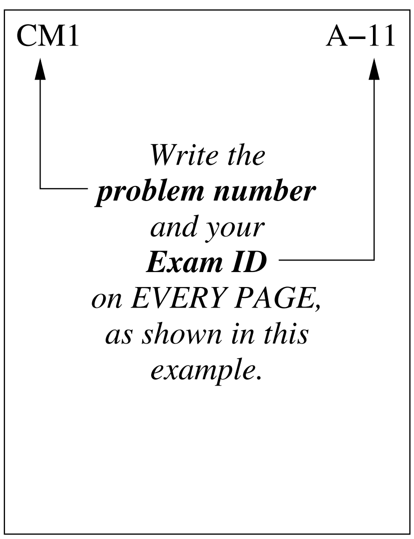
Day 2
Quantum Mechanics



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(QM1) A particle in some system can be in two eigenstates, with energies $E_1,\ E_2$ and the corresponding wavefunctions

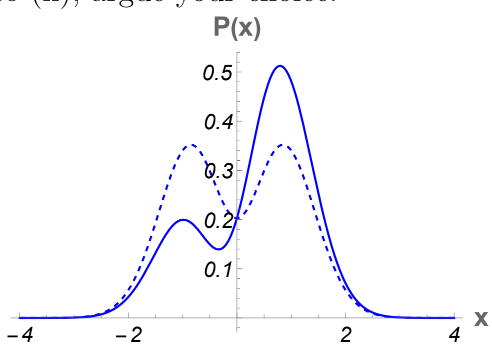
$$\psi_1(x) = \frac{1}{\pi^{1/4}}\, e^{-x^2/2} \qquad \text{and} \qquad \psi_2(x) = \frac{1}{(\pi/4)^{1/4}}\, x\, e^{-x^2/2}$$

where $x \in (-\infty, +\infty)$ is the dimensionless position.

One prepares 10 000 particles, each in an identical state $\psi(x)$, measures their energies and finds that about 3 600 of these particles have energy E_1 and the rest have energy E_2. From another experiment, it is determined that the relative phase between these energy states is $\theta = \pi/3$. We will neglect time evolution of the state, i.e. we'll keep θ fixed to this value.

We are interested in measurement of a particle's position in this system. Consider two cases: (i) the position is measured *before* the energy is measured; and (ii) the position is measured *after* the energy is measured.

(a) Below is the plot of probability density $P(x)$ of finding a particle in position x. Identify which graph (solid or dashed) corresponds to case (i) and which to (ii), argue your choice.



(b) Find the expectation value of the position in each of the two cases.

Useful integrals (integer $n = 0, 1, 2\ldots$):

$$\int_{-\infty}^{+\infty} dx\, x^{2n} e^{-x^2} = \sqrt{\pi}\, \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2^n} \qquad (= \sqrt{\pi} \ \text{ if } n = 0)$$

Solution:

According to rules of quantum mechanics a particle is in a superposition of ψ_1 and ψ_2 states before energy is measured. The energy measurement 'collapses' the state to either ψ_1 or ψ_2 with probabilities given by the coefficients of the original superposition.

The wavefunction for the superposition state of a particle is

$$\psi(x) = \sqrt{p_1}\;\psi_1(x) + e^{i\theta}\sqrt{p_2}\;\psi_2(x)$$

where $p_{1,2}$ are the probabilities to find the particle in eigenstates with energies $E_{1,2}$, and we have to include the relative phase between these. From the provided numbers we have $\theta = \pi/3$ ($\cos\theta = 1/2$) and

$$\sqrt{p_1} = \sqrt{\frac{3600}{10000}} = \sqrt{\frac{9}{25}} = \frac{3}{5} \qquad , \qquad \sqrt{p_2} = \sqrt{\frac{6400}{10000}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

Probability density to find a particle at position x in the first case is given directly by the square magnitude of the total wavefunction:

$$P_{(i)}(x) = |\psi(x)|^2 = p_1\psi_1(x)^2 + p_2\psi_2(x)^2 + 2\cos\theta\sqrt{p_1p_2}\psi_1(x)\psi_2(x)$$

In the second case we get a sum of distinguishable states: in $p_1 = 36\%$ of measurements the particle will be in state 1 with density $\psi_1(x)^2$, and in $p_2 = 64\%$ of measurements the particle will be in state 2 with density $\psi_2(x)^2$. Thus the overall probability density is

$$P_{(ii)}(x) = p_1|\psi_1(x)|^2 + p_2|\psi_2(x)|^2$$

The two differ by the interference term $\propto \psi_1(x)\psi_2(x)$. While $\psi_1(x)^2$ and $\psi_2(x)^2$ are symmetric under $x \to -x$, the interference term changes sign under $x \to -x$, which means

$$\begin{aligned} P_{(ii)}(-x) &= P_{(ii)}(x) \qquad &&\text{- dashed curve (mirror-reflection symmetric)} \\ P_{(i)}(-x) &\neq P_{(i)}(x) \qquad &&\text{- solid curve (mirror-reflection asymmetric)} \end{aligned}$$

The expectation value of the position is found by calculating first moment of the distribution

$$\langle x \rangle = \int\limits_{-\infty}^{+\infty} dx\, x\, P(x)$$

Using the provided formula we compute various integrals involving powers of x and combinations of the wavefunctions:

$$I^1_{11} \equiv \int\limits_{-\infty}^{+\infty} dx\, x\, \psi_1(x)^2 = 0 \qquad I^1_{22} \equiv \int\limits_{-\infty}^{+\infty} dx\, x\, \psi_2(x)^2 = 0 \qquad I^1_{12} \equiv \int\limits_{-\infty}^{+\infty} dx\, x\, \psi_1(x)\psi_2(x) = \frac{1}{\sqrt{2}}$$

Notice, we get zeros for two out of three integrals due to $x \to -x$ asymmetry of the integrand.

For expectation value of particle's position we get (use $2\cos\theta = 1$)

$$\langle x \rangle_{(i)} = p_1 I^1_{11} + p_2 I^1_{22} + 2\cos\theta \sqrt{p_1 p_2} I^1_{12} = \sqrt{\frac{p_1 p_2}{2}} = \frac{\sqrt{72}}{25} \approx 0.34$$
$$\langle x \rangle_{(ii)} = p_1 I^1_{11} + p_2 I^1_{22} = 0$$

As an extra exercise, you might want to calculate the variance of the position measurement. For that you will need integrals

$$I^2_{11} \equiv \int\limits_{-\infty}^{+\infty} dx\, x^2\, \psi_1(x)^2 = \frac{1}{2} \qquad I^2_{22} \equiv \int\limits_{-\infty}^{+\infty} dx\, x^2\, \psi_2(x)^2 = \frac{3}{2} \qquad I^2_{12} \equiv \int\limits_{-\infty}^{+\infty} dx\, x^2\, \psi_1(x)\psi_2(x) = 0$$

to calculate the second moment of the distribution:

$$\langle x^2 \rangle = \int\limits_{-\infty}^{+\infty} dx\, x^2\, P(x)\,, \qquad \langle x^2 \rangle_{(i)} = \langle x^2 \rangle_{(ii)} = p_1 I^2_{11} + p_2 I^2_{22} = p_1 \frac{1}{2} + p_2 \frac{3}{2}$$

The variance of x is $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ an for the two cases we get

$$\Delta x_{(i)} = \sqrt{p_1 \frac{1}{2} + p_2 \frac{3}{2} - p_1 p_2 \frac{1}{2}} = \sqrt{\frac{1281}{1250}} \approx 1.01$$
$$\Delta x_{(ii)} = \sqrt{p_1 \frac{1}{2} + p_2 \frac{3}{2}} = \sqrt{\frac{57}{50}} \approx 1.07$$

(QM2) Let $|\uparrow\rangle$ and $|\downarrow\rangle$ be eigenstates of an operator $\hat{S}_z$ with eigenvalues $\pm\frac{1}{2}\hbar$. The spin operators satisfy the commutation relations

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z, \quad [\hat{S}_y, \hat{S}_z] = i\hbar\hat{S}_x, \quad [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y. \tag{2}$$

(a) Define $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$. Find the commutation relations $[\hat{S}_z, \hat{S}_\pm]$.

(b) Show that $\hat{S}_+|\downarrow\rangle$ is an eigenstate of $\hat{S}_z$. What is the eigenvalue?

(c) Starting with $\hat{S}_+|\uparrow\rangle = 0$, find $\hat{\mathbf{S}}^2|\uparrow\rangle$. (Hint: relate $\hat{\mathbf{S}}^2 = \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ to $\hat{S}_-\hat{S}_+$)

Solution:

(a).

$$[\hat{S}_z, \hat{S}_\pm] = [\hat{S}_z, \hat{S}_x] \pm i[\hat{S}_z, \hat{S}_y] = i\hbar\hat{S}_y \pm i(-i\hbar\hat{S}_x) = \pm\hbar\hat{S}_\pm. \tag{3}$$

(b).

$$\begin{aligned}\hat{S}_z\hat{S}_+|\downarrow\rangle =& (\hat{S}_+\hat{S}_z + [\hat{S}_z, \hat{S}_+])|\downarrow\rangle \\ =& (-\frac{\hbar}{2} + \hbar)\left(\hat{S}_+|\downarrow\rangle\right) = \frac{\hbar}{2}\left(\hat{S}_+|\downarrow\rangle\right),\end{aligned} \tag{4}$$

which means $\left(\hat{S}_+|\downarrow\rangle\right)$ is an eigenstate of $\hat{S}_z$ with eigenvalue of $\hbar/2$. One may identify $\hat{S}_+|\downarrow\rangle = \hbar|\uparrow\rangle$, or $\hat{S}_+$ as a raising operator.

(c).

$$\begin{aligned}\hat{S}_-\hat{S}_+ =& (\hat{S}_x - i\hat{S}_y)(\hat{S}_x + i\hat{S}_y) \\ =& \hat{S}_x^2 + \hat{S}_y^2 + i[\hat{S}_x, \hat{S}_y] \\ =& \hat{\mathbf{S}}^2 - \hat{S}_z^2 - \hbar\hat{S}_z.\end{aligned} \tag{5}$$

Thus,

$$\hat{\mathbf{S}}^2\,|\uparrow\rangle = \left(\hat{S}_-\hat{S}_+ + \hat{S}_z^2 + \hbar\hat{S}_z\right)|\uparrow\rangle = 0 + \left(\frac{\hbar^2}{4} + \frac{\hbar^2}{2}\right)|\uparrow\rangle \tag{6}$$

which simplifies to

$$\hat{\boldsymbol{S}}^2|\uparrow\rangle = \frac{3}{4}\hbar^2|\uparrow\rangle. \tag{7}$$

This means the spin magnitude squared of a spin-1/2 particle is $(3/4)\hbar^2$.

(QM3) A particle of mass m is in a square well potential with infinite walls at $x = 0$, $x = a$

$$V(x) \;=\; \begin{cases} 0 & , \quad 0 \;<\; x \;<\; a \\ \infty & , \quad \text{otherwise} \end{cases} \tag{8}$$

The energies and eigenfunctions are

$$E_n \;=\; \frac{\hbar^2\pi^2}{2ma^2}\,n^2 \quad , \quad \phi_n(x) \;=\; \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\,n\right) \quad , \quad n \;=\; 1,\,2,\,\ldots \tag{9}$$

The particle is subject to a perturbation

$$V'(x) \;=\; -\alpha\,\delta(x - a/3) \;\; , \tag{10}$$

where $\delta(z)$ is the Dirac delta function and $\alpha > 0$ is real. The perturbation is small in the sense that $\alpha \ll aE_1$.

(a) Compute the **first order** energy correction to the lowest two eigenstates

(b) Using **only the lowest two eigenstates**, write down the **first order** correction to the **ground state** wave function. Use this to write the ground state wave function as an **explicit function** of x. Ensure that your wave function is normalized to first order in the perturbation.

(c) Use the approximate wave function from (b) to compute the probability of finding the ground state particle within the left half of the well $0 < x < a/2$. You may express the result to **first order in the perturbation**.

The following anti-derivative may prove useful

$$\int \sin(bx)\,\sin(cx)\,dx \;=\; \frac{\sin[\,(b-c)x\,]}{2(b-c)} \;-\; \frac{\sin[\,(b+c)x\,]}{2(b+c)}$$

Solution:

(a) The general expression for first order perturbation to the energy is

$$E_n^{(1)} \;=\; \left\langle\phi_n^{(0)}(x)\middle|\,V'(x)\,\middle|\phi_n^{(0)}(x)\right\rangle \tag{11}$$

$$=\; -\frac{2\alpha}{a}\int_0^a \sin^2\left(\frac{\pi x}{a}\,n\right)\,\delta(x - a/3)\,dx \;=\; -\frac{2\alpha}{a}\,\sin^2\left(\frac{\pi}{3}\,n\right) \quad .$$

Since $\sin(\pi/3) = \sin(2\pi/3) = \sqrt{3}/2$ the lowest two corrections are

$$E_1^{(1)} \;=\; E_2^{(1)} \;=\; -\frac{3\alpha}{2a} \tag{12}$$

(b) The perturbed ground state is

$$\begin{aligned}
\phi_1^{(1)}(x) &= \frac{1}{E_1^{(0)} - E_2^{(0)}} \left\langle \phi_2^{(0)}(x) \middle| V'(x) \middle| \phi_1^{(0)}(x) \right\rangle \phi_2^{(0)}(x) \qquad (13) \\
&= -\frac{\alpha\,(2/a)}{E_1^{(0)} - E_2^{(0)}} \left[\int_0^a \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi x}{a}\right) \delta(x - a/3)\, dx \right] \phi_2^{(0)}(x) \\
&= \frac{2\alpha}{3a\, E_1^{(0)}} \underbrace{\sin\left(\frac{\pi}{3}\right) \sin\left(\frac{2\pi}{3}\right)}_{=3/4} \phi_2^{(0)}(x) \;=\; \underbrace{\frac{\alpha\, a\, m}{\hbar^2\, \pi^2}}_{\alpha/2aE_1^{(0)}} \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)
\end{aligned}$$

Adding this to the unperturbed ground state gives

$$\phi_1(x) \;\simeq\; \phi_1^{(0)} + \phi_1^{(1)} \;=\; \sqrt{\frac{2}{a}} \left[\sin\left(\frac{\pi x}{a}\right) + \frac{\alpha\, a\, m}{\hbar^2\, \pi^2} \sin\left(\frac{2\pi x}{a}\right) \right] . \tag{14}$$

Since the terms are orthogonal, the second term enters $\langle \phi_1 | \phi_1 \rangle$, only at $\sim \alpha^2$. Function (14) is therefore normalized to first order.

(c) The probability density, to first order is

$$\begin{aligned}
\phi_1^2(x) &= \frac{2}{a} \left[\sin\left(\frac{\pi x}{a}\right) + \frac{\alpha\, a\, m}{\hbar^2\, \pi^2} \sin\left(\frac{2\pi x}{a}\right) \right]^2 \qquad (15) \\
&= \frac{2}{a} \left[\sin^2\left(\frac{\pi x}{a}\right) + \frac{2\alpha\, a\, m}{\hbar^2\, \pi^2} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) + \dots \right]
\end{aligned}$$

So the net probability is

$$P_{\text{left}} = \int_0^{a/2} \phi_1^2(x)\,dx = \frac{2}{a} \overbrace{\int_0^{a/2} \sin^2\left(\frac{\pi x}{a}\right)\,dx}^{=a/4} \tag{16}$$

$$+ \frac{4\alpha\, m}{\hbar^2\,\pi^2} \underbrace{\int_0^{a/2} \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right)\,dx}_{=2a/3\pi} = \frac{1}{2} + \frac{8\alpha\, a\, m}{3\hbar^2\,\pi^3}$$

This result agrees with expectations in several ways. The leading factor of $1/2$ is the probability of finding the unperturbed particle (in a simple box) in the left half. The next term shows that an attractive well ($\alpha > 0$) in the left will draw the particle into that half. This term, $(4/3)\alpha/aE_1 \ll 1$, is small because the attractive perturbation is small.