Fun in trig

(a) Sum the series

$$\cos 1^\circ \cos 2^\circ + \cos 2^\circ \cos 3^\circ + \cdots + \cos 88^\circ \cos 89^\circ =$$

(b) Unrelated to the series, compute (on a calculator):

$$\tan \frac{1^\circ}{555555} = ...$$

- in the denominator you can use any number of 5's, the more the better. You should get an interesting number. Is it a coincidence, can you explain it?

Answer of problem **Fun in trig**

(a) The series

$$S = \cos 1^\circ \cos 2^\circ + \cos 2^\circ \cos 3^\circ + \cdots + \cos 88^\circ \cos 89^\circ$$

is the same as

$$S = \sin 89^\circ \sin 88^\circ + \sin 88^\circ \sin 87^\circ + \cdots + \sin 2^\circ \sin 1^\circ$$

Add the two, and combine elements of the two series into 88 pairs:

$$\cos\alpha\cos(\alpha + 1^\circ) + \sin\alpha\sin(\alpha + 1^\circ) = \cos 1^\circ$$

So we have

$$2S = 88\cos 1^\circ \qquad \Rightarrow \qquad \boxed{S = 44\cos 1^\circ}$$

(b) In radians

$$\frac{1^\circ}{555555} = \frac{\pi}{180 * 555555}$$

and the product 180*555555 = 99999900 - number of 9's is equal to number of 5's. This number we can write as (take n - to be number of 5's in the original denominator)

$$\underbrace{99\ldots9}_{n}00 = 10^{n+2} - 100 = 10^{n+2}(1 - 10^{-n})$$

and thus for small arguments

$$\boxed{\tan\frac{1^\circ}{\underbrace{55\ldots55}_{n}} \approx \tan\frac{\pi}{10^{n+2}} \approx \pi * 10^{-(n+2)}}$$

- the more 5's, the better this approximation is! As an additional excersise determine the error that we are making as a function of n.