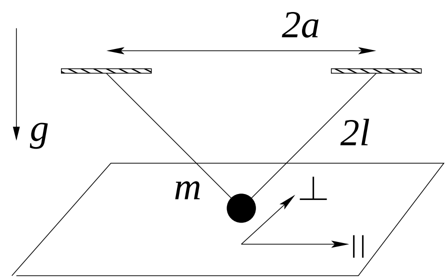
Problem of the Week

Bead on a String



An ideal (massless and unstretchable) string of length $2l$ is attached to two points separated by distance $2a$. A heavy small bead of mass m can slide along the string without friction. Gravity g is down.

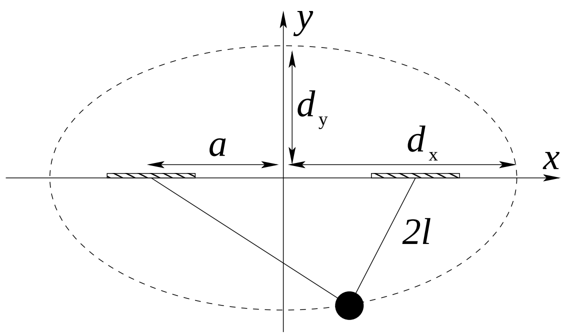
Find the small oscillation frequencies of the bead in the vertical plane perpendicular to the line connecting support points, $\omega_\perp$, and in the vertical plane that goes through the support points, $\omega_\parallel$.

Find the ratio a/l for which the projection of the bead's trajectory on the horizontal plane resembles figure "∞".

Answer of problem **Bead on a String**

In the perpendicilar plane the frequency is that of ideal pendulum with arm's length $\sqrt{l^2 - a^2}$,

$$\boxed{\omega_\perp^2 = \frac{g}{\sqrt{l^2 - a^2}} = \frac{g}{l\sqrt{1 - a^2/l^2}}}$$



In the parallel plane the bead moves on an ellipse given by

$$\frac{x^2}{d_x^2} + \frac{y^2}{d_y^2} = 1$$

with semi-axes $d_x = l$ and $d_y = \sqrt{l^2 - a^2}$. Near the equilibrium point the potential energy is mgy with

$$y = -d_y\sqrt{1 - \frac{x^2}{d_x^2}} \approx -d_y + d_y\frac{x^2}{2d_x^2}$$

where

$$R = \frac{d_x^2}{d_y}$$

is the curvature radius at the bottom of the ellipse. The frequency of small oscillations around that point is

$$\boxed{\omega_{||}^2 = \frac{gd_y}{d_x^2} = \frac{g}{R} = \frac{g}{l}\sqrt{1 - \frac{a^2}{l^2}}}$$

To get "∞" in the horizontal plane we need $\omega_\perp/\omega_{||} = 2$ which leads to

$$\boxed{\frac{a}{l} = \frac{\sqrt{3}}{2}}$$