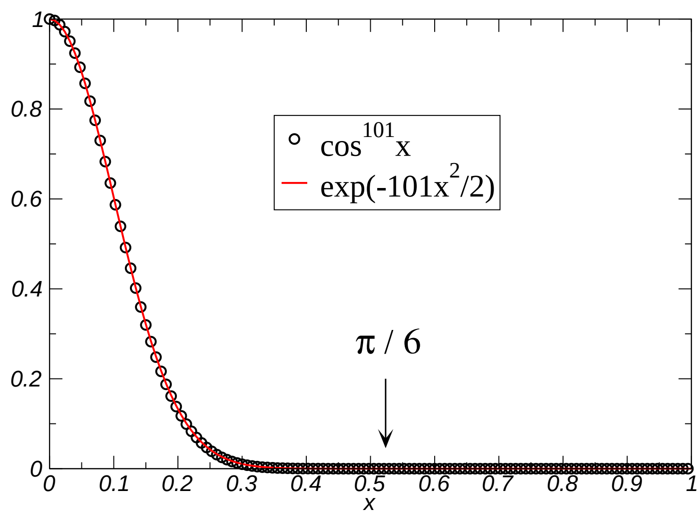
Sin 101

Without computer or Math tables find numerical value, with no less than 5% accuracy:

$$\int_{\pi/6}^{\pi/2} \sin^{101} x \, dx$$

The main contribution to this integral comes from vicinity of $x = \pi/2$, so we make sub $x = \pi/2 - x$ and write $\cos^{101} x = \exp[101 \ln(\cos x)]$. Here we can expand cos near $x = 0$, and keep only the first two terms. This expansion will be a sharply peaked function near $x = 0$, very much resembling the original function, see fugure.

$$\int_{\pi/6}^{\pi/2} \sin^{101} x \, dx = \int_0^{\pi/3} \cos^{101} x \, dx = \int_0^{\pi/3} e^{101 \ln(\cos x)} \, dx \approx \int_0^{\pi/3} e^{101 \ln(1-x^2/2)} \, dx \approx \int_0^{\pi/3} e^{-101x^2/2} \, dx$$



After that the integration is trivial: we can take the upper limit to infinity and get:

$$\int_{\pi/6}^{\pi/2} \sin^{101} x \, dx \approx \int_0^{\infty} e^{-101x^2/2} \, dx = \boxed{\sqrt{\frac{\pi}{2 \cdot 101}} \approx 0.12471}$$

Maple integration gives approximately 0.12440, difference only in forth significant digit, so we are much better than 5%!

Note that the value of the power is not an issue here, it can be even, odd, non-integer, etc as long as it is BIG. Moreover, now you can write an *asymptotic* (*i.e.* $k \to \infty$) behavior of the function

$$f(k) = \int_0^{\pi/2} \sin^k x \, dx \approx \sqrt{\frac{\pi}{2k}} \qquad k \gg 1$$