Fun in trig

(a) Sum the series

\[ \cos 1^\circ \cos 2^\circ + \cos 2^\circ \cos 3^\circ + \cdots + \cos 88^\circ \cos 89^\circ = \]

(b) Unrelated to the series, compute (on a calculator):

\[ \tan \frac{1^\circ}{555555} = \ldots \]

- in the denominator you can use any number of 5’s, the more the better. You should get an interesting number. Is it a coincidence, can you explain it?
(a) The series

\[ S = \cos 1^\circ \cos 2^\circ + \cos 2^\circ \cos 3^\circ + \cdots + \cos 88^\circ \cos 89^\circ \]

is the same as

\[ S = \sin 89^\circ \sin 88^\circ + \sin 88^\circ \sin 87^\circ + \cdots + \sin 2^\circ \sin 1^\circ \]

Add the two, and combine elements of the two series into 88 pairs:

\[ \cos \alpha \cos(\alpha + 1^\circ) + \sin \alpha \sin(\alpha + 1^\circ) = \cos 1^\circ \]

So we have

\[ 2S = 88 \cos 1^\circ \Rightarrow S = 44 \cos 1^\circ \]

(b) In radians

\[ \frac{1^\circ}{555555} = \frac{\pi}{180 \times 555555} \]

and the product \(180 \times 555555 = 99999900\) - number of 9’s is equal to number of 5’s. This number we can write as (take \(n\) - to be number of 5’s in the original denominator)

\[ \underbrace{99 \ldots 9}_n 00 = 10^{n+2} - 100 = 10^{n+2}(1 - 10^{-n}) \]

and thus for small arguments

\[ \tan \frac{1^\circ}{\underbrace{55 \ldots 55}_n} \approx \tan \frac{\pi}{10^{n+2}} \approx \pi \times 10^{-(n+2)} \]

- the more 5’s, the better this approximation is! As an additional exercise determine the error that we are making as a function of \(n\).