Plastic Deformable Earth

A “planet” is to be manufactured from a fixed amount of material (clay, for example). What shape maximizes the gravitational acceleration at a particular point on the surface? Material is incompressible.

Will this shape be the same if we consider not the volume material, but a thin shell with constant surface density?

Submitted by Charles Kanelborg
Without loss of generality, assume that we will maximize gravitational acceleration at the origin. By symmetry, the optimal shape can be described in spherical coordinates by $R(\theta)$. The surface gravity then points in the $+z$ direction, $g = g_z \hat{z}$. A mass element $dm$ at point $(r, \theta)$ contributes the following to $g_z$:

$$dg_z = G \frac{dm \cos \theta}{r^2}$$

The optimal shape will have the property that each mass element $dm$ on the surface $r = R$ of the planet will contribute the same amount, $dg_z$, to the surface gravity at the origin. Were it not so, $g_z$ could be increased by taking material from a location of lesser $dg_z/dm$ to a location of greater $dg_z/dm$. Therefore, $R(\theta)$ obeys the equation

$$\frac{dg_z}{dm} = G \frac{\cos \theta}{R^2} = \text{constant}$$

Consequently,

$$R \propto \sqrt{\cos \theta}$$

The constant of proportionality, which has units of length, can then be chosen to set the volume of the planet.

For the shell made of ideal material this argument does not work, because on one hand you can imagine crumpling the shell, without breaking it, into a tiny point; and on the other hand, even if you are forced to maintain convex shape everywhere, you cannot move a small piece of material along the surface without breaking the surface. If you don’t break the surface and modify it slightly at one point - the entire body changes shape.