Plastic Deformable Earth

A "planet" is to be manufactured from a fixed amount of material (clay, for example). What shape maximizes the gravitational acceleration at a particular point on the surface? Material is incompressible.

Will this shape be the same if we consider not the volume material, but a thin shell with constant surface density?

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Answer of problem **Plastic Deformable Earth**

Without loss of generality, assume that we will maximize gravitational acceleration at the origin. By symmetry, the optimal shape can be described in spherical coordinates by $R(\theta)$. The surface gravity then points in the $+z$ direction, $g = g_z\hat{z}$. A mass element dm at point (r, θ) contributes the following to g_z:

$$dg_z = G\frac{dm\cos\theta}{r^2}$$

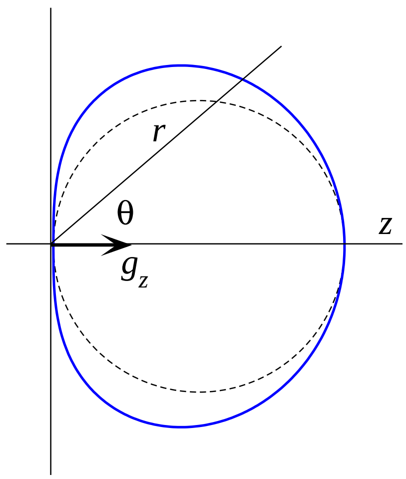

Figure 1: The shape that maximizes the gravity at the origin, $R \propto \sqrt{\cos\theta}$. The sphere, $R \propto \cos\theta$, is shown by dashed line for comparison.

The optimal shape will have the property that each mass element dm on the surface $r = R$ of the planet will contribute the same amount, dg_z, to the surface gravity at the origin. Were it not so, g_z could be increased by taking material from a location of lesser dg_z/dm to a location of greater dg_z/dm. Therefore, $R(\theta)$ obeys the equation

$$\frac{dg_z}{dm} = G\frac{cos\theta}{R^2} = constant$$

Consequently,

$$\boxed{R \propto \sqrt{\cos\theta}}$$

The constant of proportionality, which has units of length, can then be chosen to set the volume of the planet.

For the *shell* made of ideal material this argument does not work, because on one hand you can imagine crumpling the shell, without breaking it, into a tiny point; and on the other hand, even if you are forced to maintain convex shape everywhere, you cannot move a small piece of material along the surface without breaking the surface. If you don't break the surface and modify it slightly at one point - the entire body changes shape.