Condensed Matter - HW 11 :: BCS theory

PHSX 545

Problem 1

The Cooper pair wave function for a triplet state is given by rank-2 spinor

$$\psi(\mathbf{k}) = \mathbf{\Delta}(\mathbf{k}) \cdot [i\boldsymbol{\sigma}\sigma_y]\,,$$

where $\mathbf{\Delta}(\mathbf{k})$ is the vector gap function in momentum space, and $\sigma_{x,y,z}$ are Pauli matrices. Show that the expectation value for the spin of the pair is:

$$\mathbf{S} = \langle \hat{\mathbf{S}}_1 + \hat{\mathbf{S}}_2 \rangle = i\hbar \int \frac{d^3k}{(2\pi)^3}\;\cdots \times \mathbf{\Delta}(\mathbf{k})$$

and determine the missing piece to go in place of the $\ldots$.

Problem 2

The mean-field Hamiltonian in the BCS theory can be written as:

$$\mathcal{H} = E_0^{mf} + \sum_{\mathbf{k}} h_{\mathbf{k}} \qquad\qquad h_{\mathbf{k}} = \xi_{\mathbf{k}}(a^\dagger_{\mathbf{k}\uparrow}a_{\mathbf{k}\uparrow} + a^\dagger_{-\mathbf{k}\downarrow}a_{-\mathbf{k}\downarrow}) - (\Delta_{\mathbf{k}}a^\dagger_{\mathbf{k}\uparrow}a^\dagger_{-\mathbf{k}\downarrow} + \Delta^*_{\mathbf{k}}a_{-\mathbf{k}\downarrow}a_{\mathbf{k}\uparrow})$$

with $E_0^{mf} = \sum_{\mathbf{k}} \Delta^*_{\mathbf{k}}\langle a_{-\mathbf{k}\downarrow}a_{\mathbf{k}\uparrow}\rangle$.

For each pair ($\mathbf{k}\uparrow$, $-\mathbf{k}\downarrow$) consider a basis in Fock space made up of 4 states: $|\, n_{\mathbf{k}\uparrow}, n_{-\mathbf{k}\downarrow}\,\rangle = (|\,0,0\,\rangle, |\,1,0\,\rangle, |\,0,1\,\rangle, |\,1,1\,\rangle)$.

(a) By acting with $h_{\mathbf{k}}$ on $|\, n_{\mathbf{k}\uparrow}, n_{-\mathbf{k}\downarrow}\,\rangle$ show that this is a complete set of states (no new states appear). Find the eigenstates of $h_{\mathbf{k}}$ in terms of $|\, n_{\mathbf{k}\uparrow}, n_{-\mathbf{k}\downarrow}\,\rangle$ and their energies. Properly normalize them.

(b) Write the state with the lowest energy in the form $|\,BCS\,\rangle = u_{\mathbf{k}}|\,0,0\,\rangle + v_{\mathbf{k}}|\,1,1\,\rangle$. Determine $u_{\mathbf{k}}, v_{\mathbf{k}}$. Show that operator $b_{\mathbf{k}\uparrow} = u_{\mathbf{k}}a_{\mathbf{k}\uparrow} + v_{\mathbf{k}}a^\dagger_{-\mathbf{k}\downarrow}$ annihilates this state. Construct $b_{\mathbf{k}\downarrow}$ in a similar fashion.

(c) Express the other 3 eigenstates of $h_{\mathbf{k}}$ in terms of $b^\dagger$-operators acting on $|\,BCS\,\rangle$ ground state. Find the excitation energies of these states compared to the BCS ground state.